ÉRETTSÉGI VIZSGA • 2015. május 5.

MATEMATIKA ANGOL NYELVEN

EMELT SZINTŰ ÍRÁSBELI ÉRETTSÉGI VIZSGA

JAVÍTÁSI-ÉRTÉKELÉSI ÚTMUTATÓ

EMBERI ERŐFORRÁSOK MINISZTÉRIUMA

Instructions to examiners

Formal requirements:

1. Mark the paper in **ink, different in colour** from the one used by the candidate. Indicate the errors, incomplete solutions, etc in the conventional way.
2. The first one of the rectangles under each problem shows the maximum attainable score on that problem. The **points** given by the examiner are to be entered **in the rectangle** next to that.
3. **If the solution is perfect,** it is enough to enter the maximum scores in the appropriate rectangles.
4. If the solution is incomplete or incorrect, please indicate the individual **partial scores** in the body of the paper, too.

Assessment of content:

1. The markscheme may contain more than one solution for some of the problems. If the **solution by the candidate is different**, allocate the points by identifying the parts of the solution equivalent to those of the one given in the markscheme.
2. The subtotals in the markscheme can be **further divided**, but the scores awarded should always be whole numbers.
3. If it is clear that the reasoning and the final answer are both correct, you may award the maximum score even if the solution is **less detailed** than the one in the markscheme.
4. If there is a **calculation error** or inaccuracy in the solution, only take off the points for that part where the error is made. If the reasoning remains correct and the error is carried forward, the points for the rest of the solution should be awarded.
5. **In the case of a principal error,** no points should be awarded at all for that section of the solution, not even for steps that are formally correct. (These logical sections of the solutions are separated by double lines in the markscheme.) However, if the wrong information obtained owing to the principal error is carried forward to the next section or in the next part of the problem and is used correctly, the maximum score is due for the next part, provided that the error has not changed the nature of the task to be completed.
6. Where the markscheme shows a **remark** or **unit** in brackets, the solution should be considered complete without that unit as well.
7. If there are more than one different approaches to a problem, **assess only the one indicated by the candidate**.
8. **Do not give extra points** (i.e. more than the score due for the problem or part of problem).
9. **Do not take off points** for steps or calculations that contain errors but are not actually used by the candidate in the solution of the problem.
10. **Assess only four out of the five problems in Section II.** The candidate was requested to indicate in the appropriate square the number of the problem not to be assessed and counted in their total score. Should there be a solution to that problem, it does not need to be marked. However, if it is still not clear which problem the candidate does not want to be assessed, assume automatically that it is the last one in the question paper, and do not assess that problem.

I

1. **Solution 1**		
The points of tangency are the intersections of the circle k with the line perpendicular to line g through the centre.	1 point	*These 2 points are also due if these ideas are only reflected by the solution.*
The centre of circle k is the origin,	1 point	
and the equation of the line through the origin perpendicular to g is $3x - y = 0$.	2 points	
Therefore the coordinates of the points of tangency are represented by the solutions of the simultaneous equations $\left.\begin{aligned} 3x - y &= 0 \\ 4x^2 + 4y^2 &= 90 \end{aligned}\right\}$	1 point	*This point is also due if the idea is only reflected by the solution.*
From the first equation: $y = 3x$,	1 point	
By substituting in the second equation, we get: $40x^2 = 90$.	1 point	
Hence the equations have two solutions: (1.5; 4.5)	1 point	
and (–1.5; –4.5).	1 point	
(1; 3) is a normal vector of the tangents parallel to the given line.	1 point	
The equations of the tangents: $x + 3y = 15$,	1 point	
$x + 3y = -15$.	1 point	
Total:	**12 points**	

1. **Solution 2**		
The equation of a tangent in question can be expressed in the form $x + 3y = c$.	1 point	
The line $x + 3y = c$ touches the given circle if and only if the simultaneous equations below have a single solution: $\left.\begin{aligned} x + 3y &= c \\ 4x^2 + 4y^2 &= 90 \end{aligned}\right\}$	1 point	*This point is also due if the idea is only reflected by the solution.*
From the first equation: $x = c - 3y$	1 point	$y = \frac{c - x}{3}$
Substituted in the second equation: $4(c - 3y)^2 + 4y^2 = 90$.	1 point	$4x^2 + 4\left(\frac{c - x}{3}\right)^2 = 90$
Squared and rearranged: $40y^2 - 24cy + 4c^2 - 90 = 0$.	2 points	$40x^2 - 8cx + 4c^2 - 810 = 0$
There is a single solution when the discriminant is 0.	1 point	*This point is also due if the idea is only reflected by the solution.*

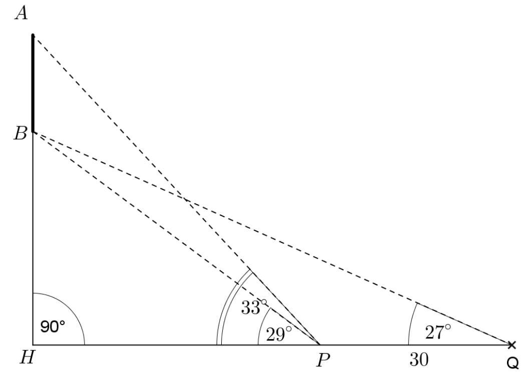
$576c^2 - 4\cdot 40\cdot(4c^2-90)=0$,	1 point	$64c^2 - 160\cdot(4c^2-810)=0$
hence $c^2=225$.	1 point	
So $c = 15$ or $c = -15$.	1 point	
The equations of the tangents: $x + 3y = 15$,	1 point	
$x + 3y = -15$.	1 point	
Total:	**12 points**	

2. a)		
I: 6	1 point	
II: 2	1 point	
III: $500\cdot 0.082$ =	1 point	
= 41 times.	1 point	
Total:	**4 points**	

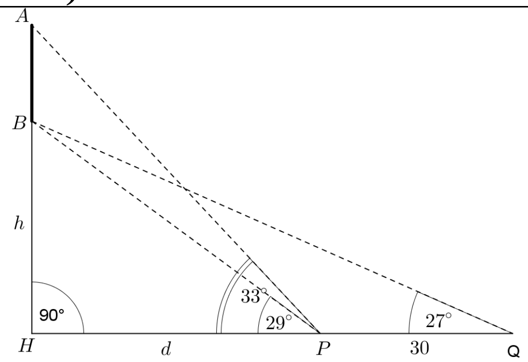
2. b)		
The number of all possible selections: $\binom{40}{10}$.	1 point	*These 2 points are also due if the idea is only reflected by the solution.*
The number of favourable cases: $\binom{8}{2}\cdot\binom{32}{8}$.	1 point	
The probability in question: $\dfrac{\binom{8}{2}\cdot\binom{32}{8}}{\binom{40}{10}}\approx$	1 point	
≈ 0.3474.	1 point	
The relative frequency is $\dfrac{0.332}{0.3474}\cdot 100\approx 95.6\%$ of the probability calculated.	1 point	
Total:	**5 points**	

Remark. Award no points for this part if the candidate uses an inappropriate model (e.g. applies the binomial distribution).

2. c)		
The probability of selecting a defective bead is $\dfrac{8}{40}=\dfrac{1}{5}=0.2$, and that of selecting a good one is 0.8.	1 point	*This point is also due if the idea is only reflected by the solution.*
The probability in question: $\binom{10}{2}\cdot 0.2^2\cdot 0.8^8\approx$	2 points	*Do not divide.*
≈ 0.302.	1 point	*Accept any other correctly rounded answer, (e.g. 0.3), too.*
Total:	**4 points**	

3. a) **Solution 1**		
 Correct diagram containing the information given in the problem. (Let *H* denote the orthogonal projection of the lookout tower onto the plane of the plain.)	2 points	*These 2 points are also due if there is no diagram but the given information is used correctly in the solution.*
With these notations ($\angle BPH = 29^{o}$, so) $\angle BPQ = 151^{o}$, ($\angle BQP = 27^{o}$, while) $\angle PBQ = 2^{o}$.	1 point	*This point is also due if the measures of the two angles are shown in a correct diagram.*
(From the sine rule applied to triangle *BPQ*:) $\frac{BP}{30} = \frac{\sin 27°}{\sin 2°}$.	2 points	
Hence $BP = 30 \cdot \frac{\sin 27°}{\sin 2°} \approx 390$ metres.	1 point	
From the right-angled triangle *BHP*, $BH = BP \cdot \sin 29°$.	1 point	
The height of the hill is about 189 metres.	1 point	
Total:	**8 points**	

Remark. The other distances are HP ≈ 341 m; PA ≈ 407 m; QB ≈ 417 m; QA ≈ 433 m.

3. a) **Solution 2**		
 Correct diagram containing the information given in the problem. (Let *H* denote the orthogonal projection of the lookout tower onto the plane of the plain.	2 points	*These 2 points are also due if there is no diagram but the given information is used correctly in the solution.*
Let $HP = d$, $HB = h$ (the height of the hill). Then $h = d \cdot \tan 29°$,	1 point	
and $h = (d+30) \cdot \tan 27°$.	1 point	

Hence $d\cdot\tan 29° = (d+30)\cdot\tan 27°$,	1 point	$\frac{h}{\tan 29°} = \frac{h}{\tan 27°} - 30$
so $d = \frac{30\cdot\tan 27°}{\tan 29° - \tan 27°}$ (≈ 341 m),	2 points	$h = \frac{30\cdot\tan 29°\cdot\tan 27°}{\tan 29° - \tan 27°}$
and the height of the hill is h $(= d\cdot\tan 29°) \approx 189$ m.	1 point	
Total:	**8 points**	

3. b) Solution 1		
In triangle ABP, $\angle BPA = 4°$ and $\angle BAP = 57°$.	1 point	
(Applying the sine rule:) $\frac{AB}{BP} = \frac{\sin 4°}{\sin 57°}$,	2 points	
thus $AB \approx 390\cdot\frac{\sin 4°}{\sin 57°}$.	1 point	
The height of the lookout tower is ≈ 32 metres.	1 point	*Accept 33, too if obtained by consistent roundings.*
Total:	**5 points**	

3. b) Solution 2		
Since $HA = d\cdot\tan 33°$,	1 point	
$HA \approx 221$ (m).	2 points	*Award 1 point for using or calculating d.*
$AB = HA - HB$	1 point	
The height of the lookout tower is ≈ 32 metres.	1 point	*Accept 33, too if obtained by consistent roundings.*
Total:	**5 points**	

Remark. Take off 1 point altogether in the entire problem if the candidate does not round an answer or rounds it incorrectly.

4. Solution 1		
The roots of the equation $4x^2 - 19x + 22 = 0$ are $x_1 = 2$, $x_2 = \frac{11}{4}$.	2 points	
Since the leading coefficient of the polynomial on the left-hand side of the inequality $4x^2 - 19x + 22 < 0$ is positive,	1 point	*This point is also due for a different correct explanation (e.g. a correct diagram).*
$A = \left]2; \frac{11}{4}\right[$.	2 points	*Award at most 1 point if the interval is closed at either end.*

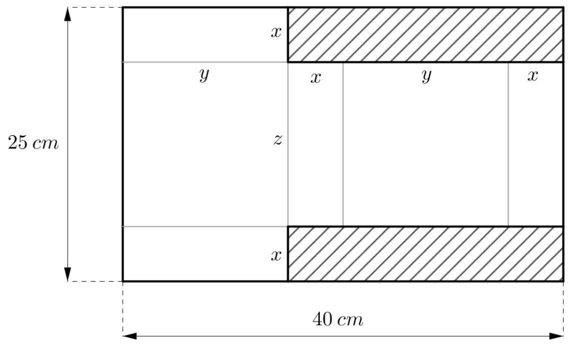
Because of $\sin 2x < 0$, $\pi + 2k\pi < 2x < 2\pi + 2k\pi$.	2 points	$2x \in \left]\pi + 2k\pi ; 2\pi + 2k\pi\right[$
$\frac{\pi}{2} + k\pi < x < \pi + k\pi$,	1 point	$B = \left]\frac{\pi}{2} + k\pi ; \pi + k\pi\right[$
where $k \in \mathbf{Z}$.	1 point	
Since $\frac{\pi}{2} < x < \pi$ for $k = 0$,	1 point	$A = \left]2; \frac{11}{4}\right[\subset \left]\frac{\pi}{2}; \pi\right[$
and since $\frac{\pi}{2} < 2$ and $\frac{11}{4} < \pi$,	2 points	$\left]\frac{\pi}{2}; \pi\right[\subset B$
it follows that $A \subset B$ holds.	1 point	
Total:	**13 points**	

Remark. Award at most 9 points if the candidate does not consider periodicity in solving the trigonometric inequality.
If the trigonometric inequality is solved in degrees, award the 4 points for solving the inequality but do not award the 4 points for investigating $A \subset B$.

4. Solution 2		
The roots of the equation $4x^2 - 19x + 22 = 0$ are $x_1 = 2$, $x_2 = \frac{11}{4}$.	2 points	
Since the leading coefficient of the polynomial on the left-hand side of the inequality $4x^2 - 19x + 22 < 0$ is positive,	1 point	*This point is also due for a different correct explanation (e.g. a correct diagram).*
$A = \left]2; \frac{11}{4}\right[$.	2 points	*Award at most 1 point if the interval is closed at either end.*
We need to prove that if $x \in \left]2; \frac{11}{4}\right[$, then $\sin 2x < 0$.	2 points	*These 2 points are also due if the idea is only reflected by the solution.*
If $2 < x < \frac{11}{4}$, then $4 < 2x < \frac{11}{2}$.	1 point	*If $x \in \left]2; \frac{11}{4}\right[$, then $2x \in \left]4; \frac{11}{2}\right[$.*
Since $\pi < 4$ and $\frac{11}{2} < 2\pi$,	2 points	$\left]4; \frac{11}{2}\right[\subset \left]\pi; 2\pi\right[$
and the values of the sine function are negative on the interval $\left]\pi; 2\pi\right[$	2 points	
it follows that the statement ($A \subset B$) is true.	1 point	
Total:	**13 points**	

II

5. a) Solution 1

 ($x = 2$ cm) One of the base edges of the cuboid is $z = 25 - 2 \cdot 2 = 21$ (cm).	1 point	
The other edge is obtained from the relationship $40 = 2x + 2y$:	1 point	*This point is also due if the idea is only reflected by the solution.*
$y = 18$ (cm).	1 point	
The surface area of the cuboid: $A(= 2 \cdot (2 \cdot 21 + 2 \cdot 18 + 21 \cdot 18)) = 912$ cm^2.	1 point	
Total:	**4 points**	

5. a) Solution 2

($x = 2$ cm) From the relationship $40 = 2x + 2y$,	1 point	*This point is also due if the idea is only reflected by the solution.*
$y = 18$ (cm).	1 point	
The other edge of the rectangle cut out is $2x + y = 22$ (cm).	1 point	
The surface area of the cuboid: $A = 40 \cdot 25 - 2 \cdot 2 \cdot 22 = 912$ cm^2.	1 point	
Total:	**4 points**	

5. b)

(Let x, y and z denote the lengths of the edges of the cuboid in cm, as shown in the diagram of part **a)**). $0 < x < 12.5$,	1 point	
$y = 20 - x$,	1 point	
and $z = 25 - 2x$.	1 point	
The volume of the cuboid: $V(x) = x \cdot (20 - x) \cdot (25 - 2x)$.	1 point	
Expanded and simplified: $V(x) = 2x^3 - 65x^2 + 500x$ ($0 < x < 12.5$).	1 point	
The derivative of the volume function V: $V'(x) = 6x^2 - 130x + 500$.	1 point	

(The function V may have a maximum or minimum where its derivative is zero.) $6x^2-130x+500=0$	1 point	
The solutions of this are $x_1 = 5$ and $x_2=\frac{50}{3}$.	1 point	
The latter is not a solution of the problem since it is not in the domain.	1 point	
The second derivative of the volume function: $V''(x)=12x-130$. $V''(5)<0$, thus there is a minimum at $x = 5$.	1 point	*This point is also due if the candidate refers to the sign change of the first derivative.*
The shorter side of the rectangle cut out is 5 cm.	1 point	
The edges of the cuboid are 5 cm, 15 cm and 15 cm, thus the maximum volume is $V(=5\cdot15\cdot15)=1125\text{ cm}^3$.	1 point	
Total:	**12 points**	

6. a)		
A 9-point tree has 8 edges.	1 point	
Therefore the sum of the degrees (i.e. the double of the number of edges) is 16.	2 points	
The sum of the given degrees is 15.	1 point	
Hence the remaining degree is 1.	1 point	
Total:	**5 points**	

Remark. Award 2 points if the candidate draws a possible tree and observes that the missing degree is 1 but does not prove that there is no other case.

6. b)		
In a simple graph on 9 points, the degree of a point may be a number 0 to 8.	2 points	
0 and 8 cannot occur together,	1 point	
that leaves only 8 possibilities, so (by the pigeonhole principle,) there must be a degree with multiplicity.	1 point	
Thus there is no simple graph on 9 point in which the degree of each point is different.	1 point	
Total:	**5 points**	

Remark. Award the maximum score of 5 points for this part if the candidate accurately refers to the theorem stating that a simple graph (on at least two points) always contains two points with the same degree.

6. c) Solution 1		
There are $\binom{9}{2}$ different ways to select two people out of 9, $\binom{7}{2}$ ways to select two out of the remaining 7, $\binom{5}{2}$ ways to select two out of the remaining 5, and $\binom{3}{2}$ ways to select two out of the remaining 3.	2 points	
If the order of the pairs counts, the number of possible selections is obtained by multiplying the results above.	1 point	*This point is also due if the idea is only reflected by the solution.*
(Since the order of the four pairs does not count,) the product needs to be divided by the number of permutations of the four pairs,	1 point	*These points are also due for a less detailed but clear reasoning.*
that is, by (4!).	1 point	
The number of possibilities is $\frac{\binom{9}{2}\cdot\binom{7}{2}\cdot\binom{5}{2}\cdot\binom{3}{2}}{4!}=945$.	1 point	
Total:	**6 points**	

6. c) Solution 2		
There are $\binom{9}{4}$ ways to select four out of the nine people.	1 point	
One out of the remaining five people needs to be paired with each of the four selected above. This can be done in $5\cdot4\cdot3\cdot2$ ways.	1 point	
The number of possible selections is obtained by multiplying the results above, if the order within the pairs counts.	1 point	*This point is also due if the idea is only reflected by the solution.*
Within each pair selected, the order of the two people may be interchanged,	1 point	*These points are also due for a less detailed but clear reasoning.*
therefore the product needs to be divided by 2^4.	1 point	
The number of possibilities is $\frac{\binom{9}{4}\cdot5\cdot4\cdot3\cdot2}{2^4}=945$.	1 point	
Total:	**6 points**	

6. c) Solution 3		
There will be one out of the nine people who has not shaken hands with anyone yet. That person can be selected in 9 ways.	2 points	
If one is selected out of the remaining 8 people, there are 7 ways to select the one who has shaken hands with him. If one is selected out of the remaining 6 people, there are 5 ways to select the one who has shaken hands with him. If one is selected out of the remaining 4 people, there are 3 ways to select the one who has shaken hands with him. The remaining 2 people have shaken hands with each other.	2 points	
The number of possible selections is obtained by multiplying the results above.	1 point	*This point is also due if the idea is only reflected by the solution.*
The number of possibilities: $9\cdot 7\cdot 5\cdot 3 = 945$.	1 point	
Total:	**6 points**	

7. Solution 1		
Let n denote the number of players taking part in the finals ($n > 1$). The scores of the players, from the last place to the first place, form a strictly increasing arithmetic sequence with first term 1 and common difference $d > 0$.	1 point	*This point is also due if the idea is only reflected by the solution.*
Since 1 point is awarded altogether in each game,	1 point	*These 2 points are also due if the idea is only reflected by the solution.*
the sum of the first n terms of the sequence (i.e. of the scores of the players) equals the number of games.	1 point	
The sum of the first n terms is $\frac{n}{2}\cdot\left(2+(n-1)\cdot d\right)$,	1 point	
the number of games is $\frac{n(n-1)}{2}$,	1 point	
therefore $\frac{n}{2}\cdot\left(2+(n-1)\cdot d\right)=\frac{n\cdot(n-1)}{2}$.	1 point	
Hence (division by $n \neq 0$) $2+(n-1)\cdot d = n-1$.	1 point	
Expanded and rearranged: $d\cdot(n-1)=n-3$.	1 pont	$2=(n-1)\cdot(1-d)$, and since $n-1>0$, it follows that $1-d>0$ also holds.
(It is known that $n \neq 1$, so) $d=\frac{n-3}{n-1}\left(=1-\frac{2}{n-1}\right)$.	1 point	
Hence $d < 1$.	1 point	
d has to be an integer multiple of 0.5,	1 point	
which is only possible if $d = 0.5$.	1 point	

In that case it follows that $n-1=4$,	1 point	
that is, 5 players took part in the finals.	1 point	
The winner scored 3 points.	1 point	
Checking: The scores of the players are 1; 1.5; 2; 2.5 and 3, which satisfy the conditions of the problem.	1 point	
Total:	**16 points**	

Remark. A possible table of scores in the finals is shown below.

	A	**B**	**C**	**D**	**E**	**points**
A		1-0	1-0	draw	draw	**3**
B	0-1		1-0	1-0	draw	**2.5**
C	0-1	0-1		1-0	1-0	**2**
D	draw	0-1	0-1		1-0	**1.5**
E	draw	draw	0-1	0-1		**1**

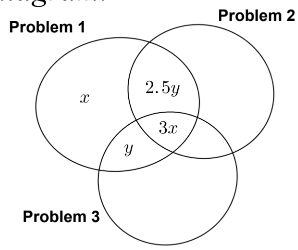
7. Solution 2		
Let n denote the number of players taking part in the finals ($n > 1$). The scores of the players, from the last place to the first place, form a strictly increasing arithmetic sequence with first term 1 and common difference $d > 0$.	1 point	*This point is also due if the idea is only reflected by the solution.*
The winner scored $1+(n-1)d$ points,	1 point	
where $1+(n-1)d \le n-1$. (Since he may have scored at most 1 point per game, he may have at most $n-1$ points.).	1 point	
Hence (because of $n > 1$) $d \le \frac{n-2}{n-1} \left(=1-\frac{1}{n-1}\right)$.	1 point	
Therefore $d < 1$.	1 point	
d has to be an integer multiple of 0.5,	1 point	
which is only possible if $d = 0.5$.	1 point	
Since 1 point is awarded altogether in each game,	1 point	*These 2 points are also due if the idea is only reflected by the solution.*
the sum of the first n terms of the sequence (i.e. of the scores of the players) equals the number of games.	1 point	
The sum of the first n terms: $\frac{2+(n-1)\cdot 0.5}{2}\cdot n$,	1 point	
the number of games: $\frac{n(n-1)}{2}$,	1 point	
therefore $\frac{2+(n-1)\cdot 0.5}{2}\cdot n=\frac{n(n-1)}{2}$.	1 point	
Hence (because of $n > 1$) $n=5$.	1 point	
There were 5 participants in the finals.	1 point	
The winner scored 3 points.	1 point	
Checking: The scores of the players are 1; 1.5; 2; 2.5 and 3, which satisfy the conditions of the problem.	1 point	
Total:	**16 points**	

8. a)		
The point of the given parabolic arc lying the farthest from the axis is the arithmetic mean of the zeros,	1 point	
that is $x = 6$.	1 point	
It can be concluded from the given diagram that the farthest point on the cubic arc is where the derivative is zero.	1 point	*This point is also due if the idea is only reflected by the solution.*
$0.03x^2 - 1.44 = 0$,	1 point	
Hence (using that $x > 0$) $x = \sqrt{48}$ (≈ 6.93).	1 point	
Total:	**5 points**	

8. b)		
(The area is obtained as the difference of the integrals of the two functions represented by the arcs.) $T = \int_0^{12} (-0.25x^2 + 3x)dx - \int_0^{12} (0.01x^3 - 1.44x)dx =$	1 point	
$= \int_0^{12} (-0.01x^3 - 0.25x^2 + 4.44x)dx =$	1 point	
$= \left[-0.0025x^4 - \frac{0.25}{3}x^3 + 2.22x^2 \right]_0^{12}$	2 points	
$T \left(= \frac{3096}{25} \right) = 123.84$.	1 point	
Total:	**5 points**	

Remark $\int_0^{12} (-0.25x^2 + 3x)dx = 72$ *and* $\int_0^{12} (0.01x^3 - 1.44x)dx = -\frac{1296}{25} = -51.84$.

8. c)		
$f(x) = \frac{-0.25x(x-12)}{0.01x(x-12)(x+12)}$	2 points	
Simplified: $f(x) = -25 \cdot \frac{1}{x+12} = g(x)$.	1 point	
The rule of assignment of the derivative function: $g'(x) = \frac{25}{(x+12)^2}$.	1 point	*Award 1 point if it is clear from a graph that the asymptotes of the hyperbola containing the graph of g are the lines y = 0 and x = –12, and another 1 point if the appropriate hyperbolic arc is drawn.*
This is positive for all $x \in]0;12[$,	1 point	
therefore the function g is strictly increasing.	1 point	
Total:	**6 points**	

9. a)		
If x denotes the number of those solving the first problem only, then the number of students solving all three problems is $3x$.	1 point	*These 2 points are also due for a correct Venn-diagram.* 
Let y be the number of those solving only the first and third problems. Then $2.5y$ solved only the first and second problems.	1 point	
From the given information: $4x + 3.5y = 22$ $3x + 2.5y = 16$	2 points	
The solution of the simultaneous equations: $x = 2,\ y = 4$.	1 point	
Thus the number of students who solved all three problems is ($3x$ =) 6.	1 point	
Checking against the wording of the problem.	1 point	
Total:	**7 points**	

9. b)		
If the mean of the grades is 3.4 then their sum is $3.4 \cdot 30 = 102$.	1 point	
Thus the sum of the six grades missing is $102 - (35 + 20 + 18 + 8 + 2) = 19$.	1 point	
If the median of the grades is 3.5 then the two middle grades in a sequence of grades listed in increasing order are a three and a four. So 15 grades are at least four and 15 grades are at most three.	1 point	*Award this point for a statement that the 15th grade is a three and the 16th grade is a four in an increasing sequence of the grades.*
(If the mode of the grades is 4 then that is the most frequently occurring grade, that is,) there are at least 3 fours among the missing grades,	1 point	
and it is not possible to have more than 3 since this completes the 15 grades that are better than three.	1 point	
Thus the other three missing grades are at most threes, and their sum is $19 - 12 = 7$.	1 point	
It is not possible to have two threes among them since that would make 3 another mode,	1 point	
(and they cannot all be worse than three,) so one of them is a three and the other two are twos.	1 point	
Therefore the six grades missing are 4, 4, 4, 3, 2, 2.	1 point	
Total:	**9 points**	*Award full mark for a clear but less detailed explanation, too.*